Eigen Value Problems in Quantum Mechanics

Stationary States

The Schrödinger equation in given by

$$i\hbar\frac{\partial\psi}{\partial t}=\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V\psi \qquad \ldots\ldots(1)$$

Let V be independent of t and it it depends only on position Co-ordinate only ,ie

$V=V(x)$. In such a case the above equation can be solved using the method of

separation of variables. Considering a product solution of the form.

$$\psi(x,t)=\psi(x).\varphi(t) \qquad \ldots\ldots\ldots\ldots(2)$$

Where $\psi(x)$ is a function that depends only on position co-ordinate x and

$\varphi(t)$ is a function of t only.

Then,

$$\frac{\partial\psi}{\partial t}=\psi\frac{\partial\varphi}{\partial t} \quad \text{and} \quad \frac{\partial^2\psi}{\partial t^2}=\psi\frac{\partial^2\varphi}{\partial t^2}$$

also, …….(3)

$$\frac{\partial\psi}{\partial x}=\varphi\frac{\partial\psi}{\partial x} \quad \text{and} \quad \frac{\partial^2\psi}{\partial x^2}=\varphi\frac{\partial^2\psi}{\partial x^2}$$

Substituting (3) in (1) we get

$$i\hbar\psi\frac{\partial\varphi}{\partial t}=\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}.\varphi+V\ \psi\varphi \qquad \ldots\ldots(4)$$

Dividing (4) by $\psi\varphi$ we get

$$i\hbar\frac{1}{\varphi}\frac{\partial\varphi}{\partial t}=\frac{-\hbar^2}{2m}\frac{1}{\psi}\frac{\partial^2\psi}{\partial x^2}+V \qquad \text{........(5)}$$

LHS of eqn. (5) is a function of t alone and RHS in a function of x alone and both are independent. Hence this equation is valid only if both sides evaluate to the same constant. Let that constant be E Then,

$$\frac{1}{\varphi}i\hbar\frac{\partial\varphi}{dt}=E \qquad \text{........(6)}$$

ie,

$$\frac{d\varphi}{dt}=\frac{-iE}{\hbar}\varphi \qquad \text{.........(7)}$$

and

$$\frac{-\hbar^2}{2m}\frac{1}{\psi}\frac{\partial^2\psi}{\partial x^2}+V=E \qquad \text{........(8)}$$

ie $$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V\psi=E\psi \qquad \text{.........(9)}$$

Thus the method of separation of variables has resulted in two ordinary differential equations.

From (7)

$$\frac{d\varphi}{\varphi}=\frac{-iE}{\hbar}dt \qquad \text{........(10)}$$

Integrating

$$\log_e \varphi = \frac{-iEt}{\hbar} + C \qquad \text{.........(11)}$$

By absorbing this constant with that of ψ we get

$$\varphi = e^{\frac{-iEt}{\hbar}} \qquad \text{..........(12)}$$

Equation (9) in called the time independent Schrödinger equation. This can be solved only if the nature of the potential V(x) is known.

Thus

$$\psi(x,t) = \psi(x)\, e^{\frac{-iEt}{\hbar}} \qquad \text{...........(13)}$$

The probability density is given by $|\psi(x,t)|^2$ and

$$|\psi(x,t)|^2 = \psi * \psi \qquad \text{.........(14)}$$

$$= \psi^*(x)\ e^{\frac{iET}{\hbar}}$$

$$= \psi^*(x)\ e^{\frac{iET}{\hbar}} . \psi(x)\, e^{\frac{-iET}{\hbar}}$$

$$= \psi^*(x)\psi(x)$$

$$|\psi(x,t)|^2 = |\psi(x)|^2 \qquad \text{...........(15)}$$

Then the probability density is independent of time. Also the expectation value is independent of time.

ie the states are stationary.

In classical Mechanics the total energy is represented by Hamiltonian operator H

$$H = KE + PE$$

$$= \frac{1}{2} m V^2 + V = \frac{m^2 v^2}{2m} + V$$

ie $$H = \frac{p^2}{2m} + V \qquad \text{..........(16)}$$

P is represented by the operator $-i\hbar \frac{\partial}{\partial x}$

$$\therefore H = \frac{(-i\hbar \frac{\partial}{\partial x})^2}{2m} + V$$

ie $$H = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V \qquad \text{...........(17)}$$

and from eqn. (9)

$$\frac{-\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi = E\psi$$

It can be written as

$$H\psi = E\psi \qquad \text{..........(18)}$$

This is time independent Schrödinger equation , ie Hamiltonian operates on ψ to return a specific value E for a particular state ψ .

A system can remain in an infinite no of stable states represented by $\psi_1(x), \psi_2(x), \psi_3(x)$..... each state having energies E_1, E_2, E_3....... ie Each allowed energy is represented by a specific wave function ψ

ie $$\psi_1(x,t) = \psi_1(x) e^{\frac{-i E_1 t}{\hbar}}$$

$$\psi_2(x,t)=\psi_2(x)e^{\frac{-iE_2t}{\hbar}}$$ and so on.

A General solution is given by a linear combination of all allowed states ie

$$-\psi(x,t)=C_1\psi_1(x,t)+C_2\psi_2(x,t)+C_3-\psi_3(x,t)+\ldots$$

$$\psi(x,t)=\sum_{n=1}^{\infty}C_n\psi_n(x)\,e^{\frac{-i\,E_nt}{\hbar}} \qquad \ldots\ldots\ldots(19)$$

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Infinite Square well

Consider a potential of the form

$V(x)=0$ if $0\leq x\leq L$

$V(n)=\infty$ otherwise

A particle trapped in such a potential is completely free except at two ends at x = 0 and x = L

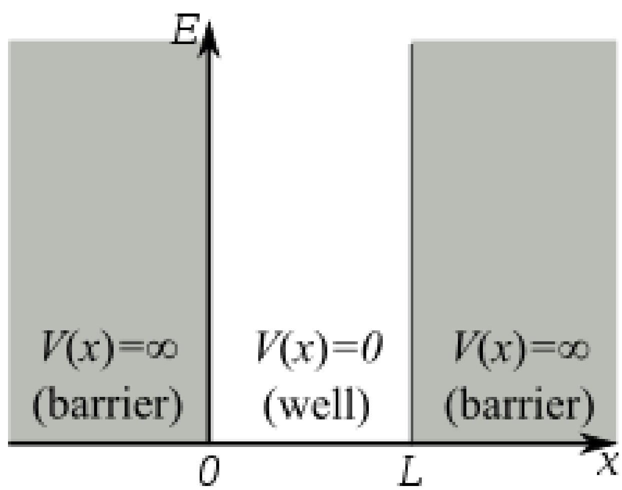

fig-1

Outside the well $\psi(x)=0$ and hence the probability of finding a particle in these region is zero.

Inside the well V(x) = 0 and Schrödinger eqn. takes the form

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}=E\psi \qquad \text{……..(1)}$$

If we substitute $k=\frac{\sqrt{2mE}}{\hbar}$ ………(2)

Multiplying eqn. (1) by $-\frac{2m}{\hbar^2}$ we get

$$\frac{d^2\psi}{dx^2}=\frac{-2mE}{\hbar^2}\psi \qquad \text{……….(3)}$$

from (2) and (3) we get

$$\frac{d^2\psi}{dx^2}=-k^2\psi \qquad \text{……….(4)}$$

$$\frac{d^2\psi}{dx^2}+k^2\psi=0 \qquad \text{……….(5)}$$

This resembles the equation for a simple harmonic oscillator given by

$$\frac{d^2x}{dt^2}+\omega^2x=0 \qquad \text{…………(6)}$$

Where solution is given by

$$x(t)=A\sin\omega t+B\,cost\,\omega t \qquad \text{…………….(7)}$$

On similar lines the solution for eqn. (5) is

$\psi(x) = A \sin kx + B \cos kx$ …………..(8)

where A and B are arbitrary constants which are decided by the boundary

conditions of the system. The function $\psi(x)$ has to be well behaved.

ie $\psi(x)$ *and* $\frac{d\psi}{dx}$ Should be continuous through out. This requires that

$\psi(0) = \psi(a) = 0$ …………(9)

ie $\psi(0) = 0 = A \sin 0 + B \cos 0$

Cos 0 = 1 hence B = 0 ……………….(10)

Hence $\psi(x) = A \sin kx$ ……………. (11)

Similarly $\psi(a) = 0$

ie $\psi(a) = 0 = A \sin ka$

if A = 0 entire $\psi(x)$ vanishes. Hence

sin ka = 0 …………..(12)

ie $ka = 0, \pm\pi, \pm 2\pi, \pm 3\pi, ...$ ………...(13)

In general

$ka = n\pi$

the (–ve) sign can be absorbed in A as $\sin(-\theta) = -\sin\theta$

or $k_n = \frac{n\pi}{a}$ …………(14)

n = 1, 2, 3………

But from eqn. (2)

$k = \frac{\sqrt{2mE}}{\hbar}$ ie $k^2 = \frac{2mE}{\hbar^2}$

$\therefore E = \frac{\hbar^2 k^2}{2m}$ …………(15)

Substituting (14) in (15) we get

$E_n = \frac{n^2 \pi^2 \hbar^2}{2\ ma^2}$ …………(16)

But $\hbar = \frac{h}{2\pi}$

Substituting this in (16) we get

$E_n = \frac{n^2 h^2}{8\ ma^2}$ ……….(17)

To find A

$\int_{-\infty}^{+\infty} \psi^* \psi \, d\tau = 1$ …………(18)

Here $\psi(x) = 0$ from $0 \leq x \leq -\infty$

$a < x < \infty$

$$\therefore \int_0^a \psi^*(x)\psi(x)\,dx = 1$$

ie $\int_0^a A\ \sin(kx)\ A\ \sin(kx)\ dx = 1$

ie $A^2\int_0^a \sin^2(kx)\,dx = 1$

$$A^2\int_0^a \left[\frac{1-\cos(2kx)}{2}\right] dx = 1$$

$$A^2\left[\int_0^a \frac{1}{2}dx - \int_0^a \frac{\cos(2kx)}{2}\,dx\right] = 1$$

$$A^2\left[\frac{1}{2}[x]_0^a - \left[\frac{\sin(2kx)}{4k}\right]_0^a\right] = 1$$

ie $A^2\frac{a}{2} = 1$

ie $A^2 = \frac{2}{a}$ $\quad A = \sqrt{\frac{2}{a}}$

$$\therefore \psi_n(x) = \sqrt{\frac{2}{a}}\ \sin kx \qquad \text{...........(19)}$$

<u>Inferences</u>

1. The wave function

$$\psi_n(x) = \sqrt{\frac{2}{a}}\ \sin\left[\frac{n\pi}{x}x\right]$$

The group of the wave function are as given below

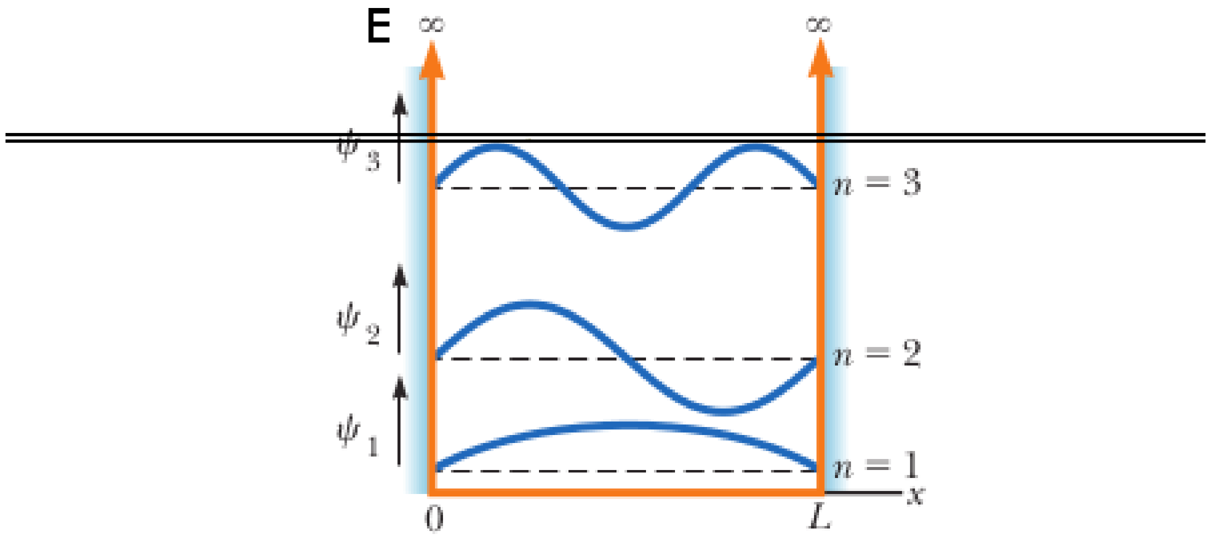

Fig-2

It is seen that with respect to the centre of the wall $\psi_1(x)$ is even, $\psi_2(x)$ is odd $\psi_3(x)$ is even and so on.Thus wave functions are alternately even and odd.

2. The points of zero crossing of wave function are called nodes. It is seen that

   $\psi_1(x)$ has no nodes, $\psi_2(x)$ has one, $\psi_3(x)$ has two and so on.

3. The wave functions are mutually orthogonal.

   ie $\int \psi_m^*(x)\psi_n(x)\, dx=0$ when $m \neq n$

$$\int \psi_m^*(x)\ \psi_n(n)\ dx=\frac{2}{a}\int_o^a \sin\left(\frac{m\pi}{a}n\right)\sin\left(\frac{n\pi x}{a}\right)dx$$

$$=\frac{2}{a}\frac{1}{2}\int_o^a \cos\left[\frac{m-n}{a}\pi x\right]-\cos\left[\frac{m+n}{a}\pi x\right]dx$$

Since

$$2\sin A\ \sin B=\cos(A-B)-\cos(A+B)$$

ie

$$\frac{1}{a}\left\{\frac{1}{(m-n)\pi}\sin\left(\frac{m-n}{a}\pi x\right)-\frac{1}{(m+n)\pi}\sin\left(\frac{m+n}{a}\pi a\right)\right\}_0^a$$

$$=\frac{1}{\pi a}\left[\frac{\sin(m-n)\pi}{m-n}-\frac{\sin(m+n)\pi}{m+n}\right]=0$$

$$\sin(m-n)\pi=\sin(m+n)\pi=\sin p\pi=0$$

Hence $\int \psi_m^*(n)\ \psi_n(x)\ dx=0$

$$\int |\psi_n(n)^2|\,dx=1$$

$$\therefore \int \psi_m^*(x)\ \psi_n(x)\ dx=\delta_{mn}$$

δ_{mn} is called Kronecker delta defined as

$$\delta_{mn}=0\ \ if\ \ m\neq n$$

$$\delta_{mn}=1\ \ if\ \ m=n$$

ie, The probability for the system file seen in two allowed states simultaneously

is zero. Thus ψ^s are ortho-normal.

4. The Energy of the particle is given by

$$E_n=\frac{n^2h^2}{8\,ma^2}$$

Hence the allowed values of energy are n = 1, 2, 3……

$$\therefore E_1=\frac{h^2}{8ma^2}\quad E_2=\frac{4h^2}{8ma^2}\quad E_3=\frac{9h^2}{8ma^2}\quad \text{etc.}$$

Thus energy values of the particle trapped in an infinite square well potential are quantized or discrete.

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The Harmonic oscillator

The classical harmonic oscillator is a mass in attached to a spring of force constant K. The eqn. of motion is given by

$$\frac{md^2x}{dt^2} = -kx$$

Ie $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$ ……..(1)

If $\frac{k}{m} = \omega^2$ $\frac{d^2x}{dt^2} + \omega^2 x = 0$ …………..(2)

The solution is $x(t) = A\sin\omega t + B\cos\omega t$, ω is the angular frequency of oscillation and x(t) is the displacement at time ***t***.

The potential $V(x) = \frac{1}{2}kx^2$

or $V(x) = \frac{1}{2}m\omega^2 x^2$ ………..(3) $\because \frac{k}{m} = \omega^2$

We have to solve the Schrödinger equation for this potential

The time independent Schrödinger equation is given by

$$\left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V\right]U = EU$$ ………………(4)

For harmonic oscillator $V(x)=\frac{1}{2}m\omega^2x^2$

$$\left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2}m\omega^2x^2\right]U=EU \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

using eqn. (5) $\frac{2m}{\hbar^2}$

$$\left[\frac{d^2}{dx^2}-\frac{m^2\omega^2}{\hbar^2}x^2+\frac{2mE}{\hbar^2}\right]U=0 \qquad \ldots\ldots\ldots\ldots\ldots(6)$$

Put $\left(\frac{m\omega}{\hbar}\right)^{1/2}=\alpha$ and $\rho=\alpha x$ …………..(7)

$\therefore \frac{d}{dx}=\frac{d}{d\rho}\frac{d\rho}{dx}$ But $\frac{d\rho}{dx}=\alpha$

$=\alpha\frac{d}{d\rho}$

$$\frac{d^2}{dx^2}=\frac{d}{dx}\frac{d}{dx}=\frac{d}{dx}\alpha\frac{d}{d\rho}=\alpha^2\frac{d^2}{d\rho^2} \qquad \ldots\ldots\ldots\ldots(8)$$

Substituting there in eqn. (6) we get

$$\left[\alpha^2\frac{d^2}{d\rho^2}-\alpha^2\rho^2+\alpha^2\lambda\right]u=0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Where $\lambda=\frac{2E}{\hbar\omega}$

ie $\left[\frac{d^2}{d\rho^2}+(\lambda-\rho^2)\right]u=0$ ……………..(10)

If $\lambda \to 0$ $\quad \frac{d^2u}{d\rho^2} - \rho^2 u = 0$ ……………………(11)

The solution for eqn. (11) can be written as

$U(p) = e^{-\frac{1}{2}\rho^2}$ ……………(12)

Thus solution for eqn. 10 can be written as

$U(p) = e^{-\frac{1}{2}\rho^2} U(\rho)$ ……………..(13)

$$\frac{dU}{d\rho} = e^{-\frac{1}{2}\rho^2} U'(\rho) - \rho\, e^{-\frac{1}{2}\rho^2} U(\rho)$$

$$\frac{d^2u}{d\rho^2} = e^{-\frac{1}{2}\rho^2} u''(\rho) + \rho e^{-\frac{1}{2}\rho^2} u'(\rho) - \rho e^{-\frac{1}{2}\rho^2} u'(\rho) - e^{-\frac{1}{2}\rho^2} u(\rho) + \rho^2 e^{-\frac{1}{2}\rho^{2}} u(\rho)$$

ie

$$\frac{d^2u}{d\rho^2} = e^{-\frac{1}{2}\rho^2} u''(\rho) - 2\rho e^{-\frac{1}{2}\rho^2} u'(\rho) + e^{-\frac{1}{2}\rho^2} u(\rho)(\rho^1 - 1)$$

Substituting in eqn. (1)

$$u''(\rho) - 2\rho\, u'(\rho) + (\rho^2 - 1 + \lambda - \rho^2) u(p) = 0$$

ie $u''(\rho) - 2\rho u'(e) + (\lambda - 1) u(\rho) = 0$ ……………(14)

It can be shown that as $\rho \to \pm\infty$ the solution for eqn.(14) diverges like

e^{ρ^2} and Hence $u(\rho)$ diverges like $e^{+\frac{1}{2}\rho^2}$

When $\lambda \to \pm\infty$, λ should take values given by

$\lambda = 2n + 1$ where n = 0, 1, 2, ……

Hence $\lambda = \frac{2E_n}{\hbar\omega} = 2n+1$

$$\therefore\ E_n = \left(n + \frac{1}{2}\right)\hbar\omega \qquad \ldots\ldots\ldots\ldots\ldots(15)$$

The solution for differential eqn. (14) $u(\rho)$ are polynomials for $\lambda = 2n+1$ and degree ***n***. It contained only even powers of ρ when ***n*** is even and is of even parity and is of odd powers of ρ when n is odd.

Parity - $\hat{p}\ \psi(x) = \psi(-x) = p\psi(x)$

$\psi(-x) = \psi(x)$ even parity

$\psi(-x) = -\psi(x)$ odd parity

These polynomials can be identified as Hermite polynomials represented by $H_n(p)$

$$H_0(\rho) = 1\ ,\ H_1(\rho) = 2\rho\ ,\ H_2(\rho) = 4\rho^2 - 2$$

Thus

$$u_n(x) = \left[\frac{\alpha}{\sqrt{\pi}\,2^n n!}\right]^{1/2} e^{-\frac{1}{2}\alpha^2 x^2} H_n(\alpha x)$$

Energy eigen values of Harmonic oscillator

We here $\lambda = 2n+1$

$$\frac{2E}{\hbar\omega}=(2n+1)$$

$$E_n=\left(n+\frac{1}{2}\right)\hbar\omega$$

n = 0, 1, 2, ...

For ground state n = 0 $E_n=\frac{1}{2}h\upsilon$

But from old quantum theory $E_n=nh\upsilon$

(Planck)

$\frac{1}{2}h\upsilon \text{ or } \frac{1}{2}\hbar\omega$ is called zero point energy. ie even in the ground state the harmonic oscillator has finite energy while classical mechanics predicted zero energy at ground state. The existence of zero point energy is in agreement with experiments and is an important feature of quantum mechanics.

Eigen Function of harmonic oscillator

$$\psi_n(x)=Ne^{-\frac{1}{2}\alpha^2x^2}H_n(\alpha x)$$

Remember $\psi_n=U_n$

The wave function corresponding to n = 0 is $H_0(\alpha n)=1$

$$\psi_n(n)=\left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2}e^{-\frac{\alpha^2x^2}{2}}$$

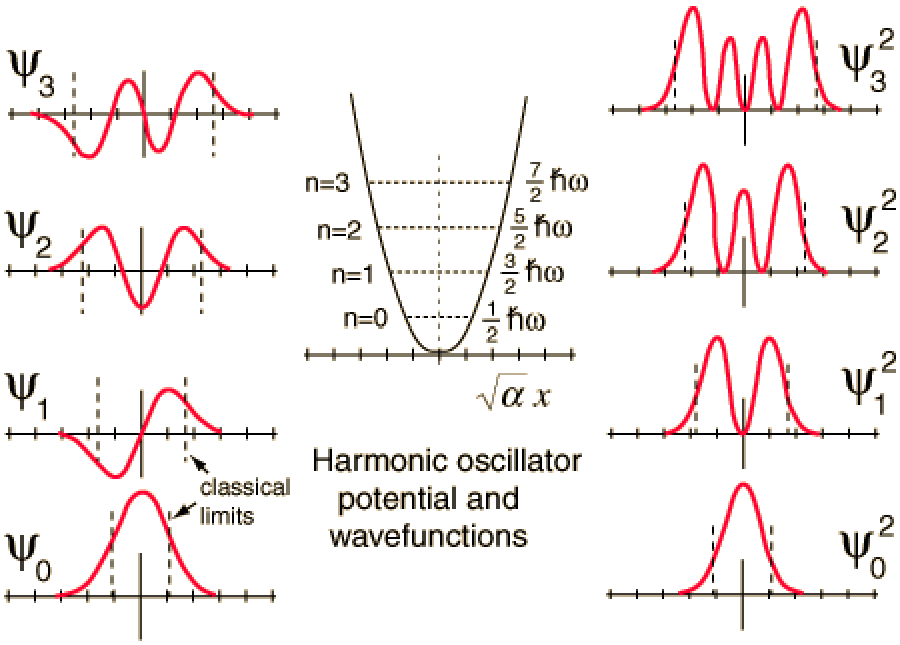

Fig-3

The probability $\psi_n^{*}\psi_n$ has a finite value beyond the classical limits. For the lowest state $E_0=\frac{1}{2}\hbar\omega$ the probability is maximum at the centre while classical theory states that maximum time in spent near the ends. For the first excited state, the particle is found to be more probable at the ends rather than at the centre.

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Simple Harmonic Oscillator

(Abstract Operator Method) :

Let us define two operators

$$\hat{a}=\left(\frac{m\omega}{2\hbar}\right)^{1/2}\hat{x}+i\left(\frac{1}{2m\omega\hbar}\right)^{1/2}\hat{p}_x$$

$$\hat{a}^+=\left(\frac{m\omega}{2\hbar}\right)^{1/2}\hat{x}-i\left(\frac{1}{2m\omega\hbar}\right)^{1/2}\hat{p}_x$$

$$[x,p_x]=i\hbar$$

$$aa^+=\left(\frac{m\omega}{2\hbar}\right)\hat{x}^2+\left(\frac{1}{2m\omega\hbar}\right)p_x^2+\frac{1}{2}$$

$$a^+a=\left(\frac{m\omega}{2\hbar}\right)\hat{x}^2+\left(\frac{1}{2m\omega\hbar}\right)\hat{p}_x^2-\frac{1}{2}$$

$$aa^+-a^+a=1$$

ie

$$[a,a^+]=\hat{I}$$

Hamiltonian Operator H = KE + PE

ie $H=\frac{p_x^2}{2m}+\frac{1}{2}m\omega^2x^2$ $\omega^2=\frac{k}{m}$

$$\frac{\hat{H}}{\hbar\omega}=\frac{p_x^2}{2m\hbar\omega}+\frac{1}{2}\frac{m\omega}{\hbar}x^2$$

$$\frac{\hat{H}}{\hbar\omega}=[a^+a+\tfrac{1}{2}]$$

$$\hat{H}=[a^+a+\tfrac{1}{2}]\hbar\omega$$

$$[a^+a, a]=a^+aa-aa^+a$$

$$=(a^+a-aa^+)a$$

But $aa^+-a^+a=1$

$$a^+a-aa^+=-1$$

ie $[a^+a, a]=-a$ ....(1)

Similarly $[a^+a, a^+]=a^+aa^+-a^+a^+a$

$$=a^+(aa^+-a^+a)$$

ie

$$[a^+a, a^+]=a^+ \qquad ....(2)$$

Let |u> be the eigen ket of operator a^+a

ie $a^+a|u> = \lambda|u>$

where λ is the eigen value.

$$a^+a\,a|u> = (aa^+-1)a|u>$$

[From (1)

$$a^+aa-aa^+a=-a$$

$$a^+ aa = aa^+ a - a = (aa^+ - 1)a]$$

ie

$$a^+ a\ a\ |u> = aa^+ a|u> - a|u> ¿$$
¿

But $a^+ a|u> = \lambda|u> ¿$
¿

ie $a^+ aa|u> = a\ \lambda|u> - a|u> ¿$
¿

ie $a^+ a\ a|u> = (\lambda - 1)\ a|u> ¿$
¿

ie $a|u> ¿$ is an eigen ket of operator $a^+ a$ belonging to eigen value $(\lambda-1)$.

Operating by $\hat{a}$ decreases the eigen value by 1 unit.

Similarly $a^+ a\ a^2|u> = (\lambda - 2)\ a^2|u> ¿$
¿

Also $a^+ aa^+|u> = (a^+ + a^+ a^+ a)|u> ¿$
¿

since

$$a^+ aa^+ - a^+ a^+ a = a^+$$

$$a^+ aa^+ = a^+ + a^+ a^+ a$$

ie

$$a^+ aa^+|u> = a^+|u> + a^+ a^+ a|u> ¿$$
¿

$$a^+ a|u> = \lambda|u> ¿$$
¿

$$a^+ aa^+|u> = a^+ u + a^+ \lambda|u> ¿$$
¿

$a^+ aa^+|u>=(\lambda+1)a^+|u>¿$
¿

ie $a^+|u>$ is also are eigen function of $\varphi\, a^+a$ belonging to eigen value $\lambda+1$

Similarly $a^+ aa^{+2}|u>=(\lambda+2)a^{+2}|u>¿$
¿

Application $\hat{a}$ decreases the eigen value by 1 and a^+ increases the eigen value by 1.

Thus $\hat{a}$ and a^+ are called ladder operators.

But $H=\left(a^+ a+\frac{1}{2}\right)\hbar\omega$

also, $E=\left(n+\frac{1}{2}\right)\hbar\omega$

Thus a^+a corresponds to the number operator which learns the present energy state of the system Application of $\hat{a}$ and a^+ decreases and increases energy eigen values in units of $\hbar\omega$. Thus a^+ creates one quanta of energy and $\hat{a}$ destroys one quanta of energy $\hat{a}$ and a^+ can be called as annihilation and creation operators.

Let U_0 be the ground state wave function of harmonic oscillator. Application of $\hat{a}$ on U_0 gives zero because the system has reached minimum energy state.

ie a $U_0 = 0$

ie $\left[\left(\frac{m\omega}{2\hbar}\right)^{1/2}\hat{x}+i\frac{1}{(2m\omega\hbar)^{1/2}}\hat{p}\,x\right]U_0=0$

But $\hat{p}_x = -i\hbar \frac{\partial}{\partial x}$

ie $\left[\sqrt{\frac{m\omega}{2\hbar}}x + i\,\frac{1}{\sqrt{2m\omega\hbar}} - i\hbar\frac{d}{dx}\right]U_0 = 0$

ie $\left[\sqrt{\frac{m\omega}{2\hbar}}x + \frac{\hbar}{\sqrt{2m\omega\hbar}}\frac{d}{dx}\right]U_0 = 0$

ie $\left[\sqrt{\frac{m\omega}{2\hbar}}x + \sqrt{\frac{\hbar}{2m\omega}}\,\frac{d}{dx}\right]U_0 = 0$

$$\sqrt{\frac{\hbar}{2m\omega}}\left[\frac{d}{dx} + \sqrt{\frac{m\omega}{2\hbar}.\frac{2m\omega}{\hbar}}x\right]U_o = 0$$

Since $\sqrt{\frac{\hbar}{2m\omega}} \neq 0$

$$\left[\frac{d}{dx} + \sqrt{\frac{m^2\omega^2}{\hbar^2}}x\right]U_0 = 0$$

$$\frac{dU_0}{dx} + \frac{m\omega}{\hbar}x\;U_0 = 0$$

$$\frac{dU_0}{dx} = -\frac{m\omega}{\hbar}x\;U_0$$

ie $\frac{dU_0}{U_0} = \frac{-m\omega}{\hbar}x$

Integrating

$$\log\ U_0 = \frac{-m\omega}{\hbar^e}\ \frac{x^2}{2} + a\ (constant)$$

$$\therefore\ U_0 = Ae^{\frac{-m\omega}{2\hbar}x^2}$$

$$\frac{To\ find\ A}{normalize\ U_0}\text{-------} \rightarrow \int_{-\infty}^{+\infty} U_0^* U_0 dx = 1$$

$$\int A^* e^{\frac{-m\omega}{2\hbar}x^2}\ Ae^{\frac{-m\omega}{2\hbar}x^2}\ dx = 1$$

$$|A|^2 \int_{-\infty}^{+\infty} e^{\frac{-2m\omega}{2\hbar}x^2}\ dx = 1$$

$$2|A|^2 \int_0^\infty \exp\left[\left(\frac{-m\omega}{2\hbar}\right)x^2\right]\ dx = 1$$

Put $\frac{m\omega}{\hbar}x^2 = u$ Differentiating $\frac{m\omega}{\hbar}2xdx = dU$

But $x^2 = u . \frac{\hbar}{m\omega}$

$$x = \sqrt{\frac{\hbar}{m\omega}} . \sqrt{u} \Rightarrow \frac{m\omega}{\hbar} . \sqrt{\frac{\hbar}{m\omega}}\sqrt{u}\, 2\, dx = du$$

ie $2\sqrt{\frac{m\omega}{\hbar}}\sqrt{u}\ dx = du \Rightarrow dx = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}u^{-\frac{1}{2}}\ du$

ie $2|A|^2 \frac{1}{2}\int_0^\infty \exp(-u)\sqrt{\frac{\hbar}{m\omega}}u^{-\frac{1}{2}}\ du = 1$

$$|A|^2 2.\sqrt{\frac{\hbar}{m\omega}} \times \frac{1}{2}\int_0^\infty \exp(-u)u^{-\frac{1}{2}}\ du = 1$$

ie $|A^2|\sqrt{\frac{\hbar}{m\omega}}\int_0^\infty \exp(-u)u^{-\frac{1}{2}}\,du=1$

From standard result of gamma function

$$\sqrt{n}=\int_0^\infty e^{-y}\,y^{n-1}\,dy$$

Here $\int_0^\infty \exp(-u)\,u^{-\frac{1}{2}}\,du=\int_0^\infty \exp(-u)^{\frac{1}{u^2}-1}\,dy$

Comparing with standard result $n=\frac{1}{2}$

$$\therefore \int_0^\infty \exp(-u)\,u^{\frac{-1}{2}}\,du=\sqrt{\frac{1}{2}}=\sqrt{\pi} \qquad \text{But } \sqrt{\frac{1}{2}}=\sqrt{\pi}$$

$$\therefore \sqrt{\frac{\hbar}{m\omega}}.\sqrt{\pi}\,|A|^2=1$$

ie $\sqrt{\frac{\hbar\pi}{m\omega}}\,|A|^2=1$

ie $|A|^2=\sqrt{\frac{m\omega}{\pi\hbar}}$ ie $|A|^2=\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}$

ie $|A|=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$

$$\therefore U_0=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}.\exp\left(\frac{-m\omega}{2\hbar}\right)x^2$$

The Schrödinger equation is $H\psi_0=E\psi_0$

ie $(a^{+}a+\frac{1}{2})\hbar\omega\ \psi_0=E\ \psi_0$

The ground state energy is $E_0=\frac{1}{2}\hbar\omega$

and energy in n^{th} state in $E_n=(n+\frac{1}{2})\ \hbar\omega$

Watch Video

References:

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Simulations:

1. Q-M 2d-Box
2. Quantum Tunneling-colorado uty.